New classes of analytic solutions of the three-state problem

2000 J. Phys. A: Math. Gen. 33 5041

(http://iopscience.iop.org/0305-4470/33/28/308)

J. Phys. A: Math. Gen. **33** (2000) 5041–5047. Printed in the UK PII: S0305-4470(00)10734-6

New classes of analytic solutions of the three-state problem

A M Ishkhanyan

Engineering Center of Armenian National Academy of Sciences, Ashtarak-2, 378410 Armenia

E-mail: artur@ec.sci.am

Received 5 January 2000

Abstract. A rich multiparametric class of solutions of the time-dependent Schrödinger equations, including many integrable cases known at present of the three-state problem, is derived. In addition, the class contains an infinite number of new multiparametric integrable subfamilies. Examples of such four-parametric new subfamilies of asymmetric, in general, amplitude and frequency modulation functions permitting solution of the three-state problem in terms of the Clausen function are presented.

1. Introduction

The analytic solutions of the time-dependent Schrödinger equations have played a very important role in the establishment of a number of unexpected and challenging effects in various branches of atomic and optical physics as well as in many other fields (see, for instance [1–20] and references therein). It is for this reason that, after the seminal paper of Bambini and Berman [10] demonstrating the existence of entire classes of solvable models that may possess unusual physical properties, the systematic search for analytically integrable cases has received a great dial of attention during the last two decades. At present a number of families of analytic solutions of both two-state [20–30] and three-state [31–38] problems are reported by various authors.

However, it appears that these numerous classes of solutions are not actually independent. In fact, it is already recognized that many of them can be united into more general families containing, in addition, a variety of new classes. An illustrative example is the recent generalization of all known two-state models that are solvable in terms of confluent hypergeometric functions to a single formula [29]. Another example is the derivation of similar results for the solutions of the two-state problem in terms of hypergeometric functions [30]. These results beg the question whether it is possible to also generalize the solutions of the three-state problem. It should be noted here that the key component of the above generalizations for the two-state model are the class property of the solvable models and the approach of the equation of invariants proposed in [28, 29]. Though the usefulness of the second component, the equation of invariants, is restricted by the two-state problem only, the specific class property of the solutions of the time-dependent Schrödinger equations providing the construction of the generalized classes [28–30] is, in general, also an attribute of multi-state problems. So one may also expect similar generalizations in this case.

In this paper we present exact analytic solutions for the three-state problem derived by reduction of the initial system to a model third-order ordinary differential equation having three complex singular points. The solutions form a ten-parametric (four discrete and six continuous)

0305-4470/00/285041+07$30.00

family covering a number of previously known solutions and, moreover, suggesting a variety of new classes. Furthermore, it is possible to enlarge the class obtained to cover all the known integrable classes.

Without any concern for a specific physical context, we consider a model three-state system in which the transitions between states 1 and 2 and between states 2 and 3 are permitted and the transition between states 1 and 3 is forbidden. The time evolution of the system is governed by the Schrödinger equations for probability amplitudes a_1, a_2 and a_3:

$$\begin{aligned} \mathrm{i}\frac{\mathrm{d}a_1}{\mathrm{d}t} &= U\mathrm{e}^{-\mathrm{i}\delta_1}a_2 \\ \mathrm{i}\frac{\mathrm{d}a_2}{\mathrm{d}t} &= U\mathrm{e}^{+\mathrm{i}\delta_1}a_1 + V\mathrm{e}^{-\mathrm{i}\delta_2}a_3 \\ \mathrm{i}\frac{\mathrm{d}a_3}{\mathrm{d}t} &= V\mathrm{e}^{+\mathrm{i}\delta_2}a_2 \end{aligned} \tag{1}$$

where the amplitude (U, V) and frequency $(\delta_{1,2})$ modulations are assumed to be time-dependent functions. This system is equivalent to one third-order linear differential equation for a_1 (hereafter the alphabetical index denotes differentiation with respect to the corresponding variable):

$$\begin{aligned} a_{1ttt} &+ \left(2\mathrm{i}\delta_{1t} - 2\frac{U_t}{U} + \mathrm{i}\delta_{2t} - \frac{V_t}{V}\right)a_{1tt} \\ &+\left[\left(\mathrm{i}\delta_{1t} - \frac{U_t}{U}\right)_t + U^2 + V^2 + \left(\mathrm{i}\delta_{1t} - \frac{U_t}{U}\right)\left(\mathrm{i}\delta_{1t} - \frac{U_t}{U} + \mathrm{i}\delta_{2t} - \frac{V_t}{V}\right)\right]a_{1t} \\ &+U^2\left(\mathrm{i}\delta_{1t} + \frac{U_t}{U} + \mathrm{i}\delta_{2t} - \frac{V_t}{V}\right)a_1 = 0. \end{aligned} \tag{2}$$

Following our earlier paper [28–30], we consider the reduction of this equation via transformation of both independent and dependent variables to some third-order linear differential equation having a known analytic solution:

$$u_{zzz} + f(z)u_{zz} + g(z)u_z + h(z)u = 0. \tag{3}$$

First we consider the formal solutions of system (1) *rewritten for a complex argument* z and note that, if the functions $a^*_{1,2,3}(z)$ are a solution of this system for some $U^*(z)$, $V^*(z)$ and $\delta^*_{1,2}(z)$ (the set of these functions is referred to as the basic integrable model), then the functions $a_{1,2,3}(t) = a^*_{1,2,3}(z(t))$ are the solution of the initial system (1) for functions $U(t)$, $V(t)$ and $\delta_{1,2}(t)$ given by the simple formulae:

$$\begin{aligned} U(t) &= U^*(z)\frac{\mathrm{d}z}{\mathrm{d}t} \\ V(t) &= V^*(z)\frac{\mathrm{d}z}{\mathrm{d}t} \\ \delta_{1,2t}(t) &= \delta^*_{1,2z}(z)\frac{\mathrm{d}z}{\mathrm{d}t} \end{aligned} \tag{4}$$

for an arbitrary complex-valued function $z(t)$. The last equation of this system is simply another form of the obvious relation $\delta_{1,2}(t) = \delta^*_{1,2}(z(t))$ that, however, is written here in this form for purposes which will be explained below.

Thus, the sets of functions U, V and $\delta_{1,2}$ for which the three-state problem (1) is solvable form classes. Each integrable model of the time-dependent Schrödinger equations generates an entire infinite class, since $z(t)$ is arbitrary, of (complex) solvable cases. This simple property allows one to generate new solvable cases from the known ones, as well as to obtain a number of new integrable models with real functions $U(t)$, $V(t)$ and $\delta_{1,2}(t)$ via an appropriate choice of the free parameters available.

In order to find independent basic solvable models we consider, according to the above, only the transformation of the dependent variable:

$$u = \varphi(z) \cdot a_1(z) \tag{5}$$

that changes equation (3) into the form

$$a_{1zzz} + \left(3\frac{\varphi_z}{\varphi} + f\right) a_{1zz} + \left(3\frac{\varphi_{zz}}{\varphi} + 2f\frac{\varphi_z}{\varphi} + g\right) a_{1z} + \left(\frac{\varphi_{zzz}}{\varphi} + f\frac{\varphi_{zz}}{\varphi} + g\frac{\varphi_z}{\varphi} + h\right) a_1 = 0. \tag{6}$$

Then, comparing this equation with equation (2) rewritten for z, $U^*(z)$, $V^*(z)$, and $\delta^*_{1,2}(z)$, we get three equations for the determination of the functions U^*, V^*, $\delta^*_{1,2}$ and $\varphi(z)$. The elimination of $\varphi(z)$ from this system leads to a system of two nonlinear differential equations which presents a three-state analogue of the equation of invariants for the two-state problem [29]. Though this system presents a considerably more complicated mathematical problem than the equation of invariants, however, some solutions may be easily derived, as shown below.

2. A 10-parametric class of integrable models

Consider the case when the transformed target equation (6) has the form

$$a_{1zzz} + \frac{f_1 z + f_0}{z(1-z)} a_{1zz} + \frac{g_p(z)}{z^2(1-z)^2} a_{1z} + \frac{h_p(z)}{z^3(1-z)^3} a_1 = 0 \tag{7}$$

where $f_{0,1}$ are constants and g_p, h_p are arbitrary polynomials in z. This equation permits a general solution in the form of a superposition of power series convergent for all $|z| \leqslant 1$ because it has only simple singular points: $z = 0, 1, \infty$ (for further details and some non-essential restrictions not mentioned here see, for instance, [39]):

$$u(z) = \sum_{m=1}^{3} \left(C_m z^{\eta_m} \sum_{k=0}^{\infty} B_k^m z^k \right) \tag{8}$$

where $C_{1,2,3}$ are arbitrary constants to be defined from the initial conditions, B_k^m are determined by some recurrent relations and η_m are the roots of the following cubic characteristic equation:

$$\eta(\eta-1)(\eta-2) + f_0\eta(\eta-1) + g_0\eta + h_0 = 0 \tag{9}$$

where g_0 and h_0 are the free terms of the polynomials g_p and h_p.

The system of equations for determination of the basic integrable cases now becomes

$$\begin{aligned}
&2\mathrm{i}\delta^*_{1z} - 2\frac{U^*_z}{U^*} + \mathrm{i}\delta^*_{2z} - \frac{V^*_z}{V^*} = \frac{f_1 z + f_0}{z(1-z)} \\
&\left(\mathrm{i}\delta^*_{1z} - \frac{U^*_z}{U^*}\right)_z + U^{*2} + V^{*2} + \left(\mathrm{i}\delta^*_{1z} - \frac{U^*_z}{U^*}\right)\left(\mathrm{i}\delta^*_{1z} - \frac{U^*_z}{U^*} + \mathrm{i}\delta^*_{2z} - \frac{V^*_z}{V^*}\right) = \frac{g_p(z)}{z^2(1-z)^2} \\
&U^{*2}\left(\mathrm{i}\delta^*_{1z} + \frac{U^*_z}{U^*} + \mathrm{i}\delta^*_{2z} - \frac{V^*_z}{V^*}\right) = \frac{h_p(z)}{z^3(1-z)^3}.
\end{aligned} \tag{10}$$

It is immediately seen that this system is satisfied if we set

$$\begin{aligned}
U^*(z) &= U^*_0 z^{k_1}(1-z)^{n_1} \\
V^*(z) &= V^*_0 z^{k_2}(1-z)^{n_2} \\
\delta^*_{1,2z}(z) &= \frac{\beta_{1,2}}{z} + \frac{\gamma_{1,2}}{1-z}
\end{aligned} \tag{11}$$

with any integer or half-integer $k_{1,2}, n_{1,2} \geqslant -1$, the parameters $U^*_0, V^*_0, \beta_{1,2}, \gamma_{1,2}$ being arbitrary complex constants.

Thus, we have derived an infinite 10-parametric class of analytically integrable models of the three-state problem explicitly given by equations (4) and (11). Four of these parameters, $k_{1,2}$ and $n_{1,2}$, are discrete. Fixing these parameters we obtain subfamilies with six continuous parameters.

In the case of constant detunings, $\delta_{1,2} = \Delta_{1,2} \cdot t$, the obtained class defines the following quasi-stepwise or bell-shaped asymmetric, in general, four-parametric classes of pulses given in parametric form (we set $\beta_{1,2} = \Delta_{1,2}\mu$, $\gamma_{1,2} = \Delta_{1,2}(\lambda + \mu)$, $U_0^* = \mu U_0$, $V_0^* = \mu V_0$)

$$U = \mu U_0 \frac{z^{k_1+1}(1-z)^{n_1+1}}{\lambda z + \mu} \qquad V = \mu V_0 \frac{z^{k_2+1}(1-z)^{n_2+1}}{\lambda z + \mu} \tag{12}$$

$$\mathrm{e}^t = \frac{z^\mu}{(1-z)^{\lambda+\mu}}. \tag{13}$$

When $\lambda = 0$, the relation (13) can be inverted to get $z = 1/(1 + \mathrm{e}^{-t/\mu})$. Then the derived class is explicitly given by

$$U = \frac{U_0}{(1+\mathrm{e}^{-t/\mu})^{1+k_1}(1+\mathrm{e}^{t/\mu})^{1+n_1}} \qquad V = \frac{V_0}{(1+\mathrm{e}^{-t/\mu})^{1+k_2}(1+\mathrm{e}^{t/\mu})^{1+n_2}}. \tag{14}$$

As seen, the members of this subfamily are symmetric in time when $k_1 = n_1$ and $k_2 = n_2$.

3. Two subfamilies of models integrable in terms of Clausen functions

In a number of cases, it is possible to choose the parameters $k_{1,2}$, $n_{1,2}$ so that the solution $a_1(t)$ of the three-state problem can be written in terms of known mathematical functions such as the Bessel functions, Clausen function, other generalized hypergeometric functions, etc [40].

For example, when the polynomials g_p and h_p are of the form

$$g_p = (g_1 z + g_0)(1-z) \qquad h_p = (h_1 z + h_0)(1-z)^2 \tag{15}$$

then the initial target equation (3) (from which equation (7) is derived) is the equation satisfied by the Clausen function ${}_3F_2(p_1, p_2, p_3; q_1, q_2; z)$, the factor φ being of the form: $\varphi(z) = z^\eta$. So the final solution of the three-state problem in this case is explicitly given by

$$\begin{aligned} a_1 = C_1 z^{\eta_1} \cdot {}_3F_2(\eta_1 - \xi_1, \eta_1 - \xi_2, \eta_1 - \xi_3; 1 + \eta_1 - \eta_2, 1 + \eta_1 - \eta_3;\ z) \\ + C_2 z^{\eta_2} \cdot {}_3F_2(\eta_2 - \xi_1, \eta_2 - \xi_2, \eta_2 - \xi_3; 1 + \eta_2 - \eta_1, 1 + \eta_2 - \eta_3;\ z) \\ + C_3 z^{\eta_3} \cdot {}_3F_2(\eta_3 - \xi_1, \eta_3 - \xi_2, \eta_3 - \xi_3; 1 + \eta_3 - \eta_2, 1 + \eta_3 - \eta_2;\ z) \end{aligned} \tag{16}$$

where $\eta_{1,2,3}$ are defined by (9) and $\xi_{1,2,3}$ are the roots of the following cubic equation:

$$\xi(\xi - 1)(\xi - 2) - f_1\xi(\xi - 1) - g_1\xi - h_1 = 0. \tag{17}$$

The particular values of the parameters $k_{1,2}$ and $n_{1,2}$ for which (15) is the case, as well as the corresponding parameters $f_{0,1}$, $g_{0,1}$, $h_{0,1}$ (expressed in terms of U_0^*, V_0^* and $\delta_{1,2}^*$), can be easily found by direct substitution of (11) and (15) into (10). We present here two examples from a variety of new solutions in terms of the Clausen function that are of especial interest for numerous applications. Though the solutions are applicable for both amplitude and frequency modulations, here we restrict ourselves to the case of amplitude modulation only.

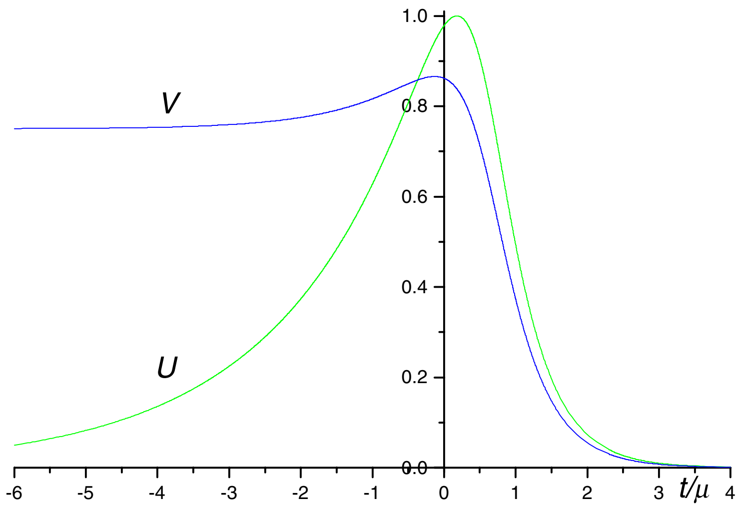

Figure 1. Pulse shapes given by equations (18), $U_0 = 1$, $V_0 = 0.75$, $\lambda/\mu = -0.75$, $\mu > 0$.

(1) The first class we present is derived when $k_1 = n_1 = -\frac{1}{2}$, $k_2 = -1$, $n_2 = -\frac{1}{2}$. One can easily check then that $\beta_1 + \beta_2 = \gamma_1 + \gamma_2 = 0$. As can be seen, this class corresponds to the two-photon resonance $\delta_1 + \delta_2 = 0$.

For constant detunings, $\delta_1 = -\delta_2 = \Delta \cdot t$, $\Delta \neq 0$, the class consists of a pair of asymmetric pulses defined by

$$U = \mu U_0 \frac{\sqrt{z(1-z)}}{\lambda z + \mu} \qquad V = \mu V_0 \frac{\sqrt{1-z}}{\lambda z + \mu} \tag{18}$$

where $z(t)$ is given by (13). The shapes of the pulses are shown in figure 1. The pulses corresponding to $\lambda = 0$ are explicitly given as

$$U = \frac{U_0}{2} \operatorname{sech}\left(\frac{t}{2\mu}\right) \qquad V = \frac{V_0}{\sqrt{1 + \mathrm{e}^{t/\mu}}}. \tag{19}$$

The parameters of the solution (16) are

$$\begin{aligned} \xi_1 &= -\tfrac{1}{2} \qquad & \xi_{2,3} &= \frac{\mathrm{i}\Delta\lambda}{2} \pm \sqrt{-\frac{\Delta^2\lambda^2}{4} + U_0^2\mu^2} \\ \eta_1 &= 0 \qquad & \eta_{2,3} &= \frac{1 - \mathrm{i}\Delta\mu}{2} \pm \mathrm{i}\sqrt{\frac{\Delta^2\mu^2}{4} + V_0^2\mu^2}. \end{aligned} \tag{20}$$

Note that the solution (16) is in force also for $\Delta = 0$, i.e., in the case of one-photon resonance, $\delta_1 = \delta_2 = 0$, though in this case, as is seen from the last equations of (4) and (11), $z(t)$ can be an arbitrary function. The resultant class of pulses has been presented by Carroll and Hioe [34]:

$$U = \frac{U_0}{\sqrt{z(1-z)}} \frac{\mathrm{d}z}{\mathrm{d}t} \qquad V = \frac{V_0}{z\sqrt{1-z}} \frac{\mathrm{d}z}{\mathrm{d}t}. \tag{21}$$

(2) The second class we present is derived when $k_1 = -1$, $n_1 = 0$, $k_2 = -1$, $n_2 = -\frac{1}{2}$. In this case one derives $\beta_1 = \gamma_1 = 0$. So that the first pulse is resonant to the transition it is related: $\delta_1 = 0$. If the second detuning is constant, $\delta_2 = \Delta \cdot t$, then the pulses of this class are given by

$$U = \mu U_0 \frac{1-z}{\lambda z + \mu} \qquad V = \mu V_0 \frac{\sqrt{1-z}}{\lambda z + \mu} \tag{22}$$

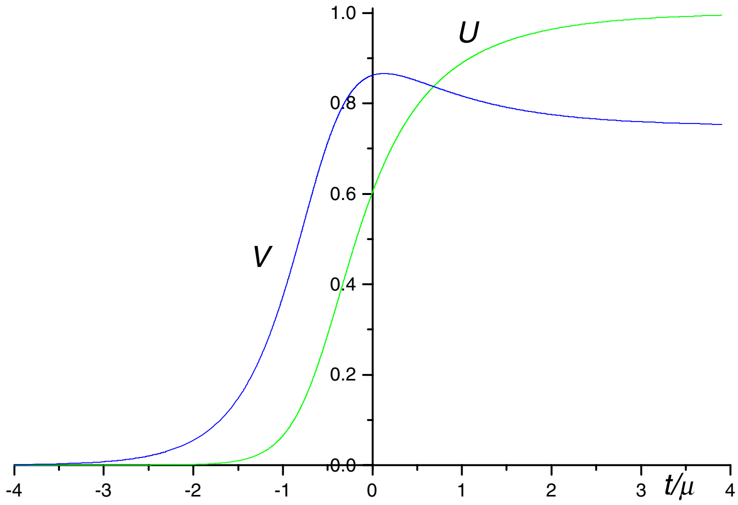

Figure 2. Pulse shapes given by equations (22), $U_0 = 1$, $V_0 = 0.75$, $\lambda/\mu = -0.75$, $\mu < 0$.

with $z(t)$ again defined by (13). These are also asymmetric pulses, in general (see figure 2). The pair corresponding to $\lambda = 0$ are

$$U = \frac{U_0}{1 + e^{t/\mu}} \qquad V = \frac{V_0}{\sqrt{1 + e^{t/\mu}}}. \tag{23}$$

The solution is again given by (16), with $\xi_{1,2,3}$ now being

$$\xi_1 = -\tfrac{1}{2} + i\Delta\mu \qquad \xi_{2,3} = \pm i U_0 \mu \tag{24}$$

and $\eta_{1,2,3}$ to be defined from the equation

$$\begin{aligned}\eta(\eta - 1)(\eta - 2) + (i\Delta\mu + 3)\eta(\eta - 1) + (i\Delta\mu + 1 + U_0^2\mu^2 + V_0^2\mu^2)\eta \\ + i\Delta\mu \cdot U_0^2\mu^2 = 0.\end{aligned} \tag{25}$$

4. Summary

In conclusion, we have presented a wide class of analytic solutions of the three-state problem that could have numerous applications in many areas of physics. The class includes a number of known solutions and additionally contains (countable) infinite number of new multiparametric subfamilies. Note that it is possible to enlarge the derived class to more general types of amplitude and frequency modulation functions. Various new interesting pulse types, including those ones having a double peak structure, will be presented elsewhere.

In a number of cases, the solution of the time-dependent Schrödinger equations can be written in terms of known mathematical functions such as the Bessel functions, Clausen function, other generalized hypergeometric functions, etc. We have presented two examples of such new subfamilies, the solutions for which are written in terms of the Clausen function.

Finally, we would like to note that the extension of the results of this paper to the multi-state case is straightforward [41].

Acknowledgments

The author is pleased to acknowledge the financial support from the NATO under Grant No CRG CRG 974301, International Science and Technology Center under Grant No A215-99 and the Armenian Ministry of Education and Science under Grant No RA 98-740.

References

[1] Rabi I I 1937 *Phys. Rev.* **51** 652
[2] Rosen N and Zener C 1932 *Phys. Rev.* **40** 502
[3] Landau L D 1932 *Phys. Z. Sowjetunion* **2** 46
Zener C 1932 *Proc. R. Soc.* A **137** 696
[4] Nikitin E E 1962 *Opt. Spektrosk.* **13** 761
[5] Demkov Y N 1963 *Zh. Eksp. Teor. Fiz.* **45** 195
[6] McCall S L and Hahn E L 1969 *Phys. Rev.* **183** 457
[7] Demkov Yu N and Kunike M 1969 *Vest. Leningr. Univ. Fiz. Khim* **16** 39
Suominen K-A and Garraway B M 1992 *Phys. Rev.* A **45** 374
[8] Allen L and Eberly J H 1975 *Optical Resonance and Two-Level Atoms* (New York: Wiley)
[9] Crothers D S F 1978 *J. Phys. B: At. Mol. Phys.* **11** 1025
[10] Bambini A and Berman P R 1981 *Phys. Rev.* A **23** 2496
[11] Hioe F T and Carroll C E 1985 *Phys. Rev.* A **30** 2100
[12] Kuklinski J R, Gaubatz U, Hioe F T and Bergmann K 1989 *Phys. Rev.* A **40** 6741
Carroll C E and Hioe F T 1990 *Phys. Rev.* A **42** 1522
[13] Oreg J, Hioe F T and Eberly J H 1984 *Phys. Rev.* A **29** 690
[14] Carroll C E and Hioe F T 1992 *Phys. Rev. Lett.* **68** 3523
[15] Hioe F T and Grobe R 1994 *Phys. Rev. Lett.* **73** 2559
[16] Grobe R, Hioe F T and Eberly J H 1994 *Phys. Rev. Lett.* **73** 3183
[17] Eberly J H, Pons M L and Haq H R 1994 *Phys. Rev. Lett.* **72** 56
[18] Hioe F T and Carroll C E 1997 *Phys. Rev.* A **56** 2292
[19] Rahman A and Eberly J H 1998 *Phys. Rev.* A **58** R805
[20] Hioe F T and Carroll C E 1985 *Phys. Rev.* A **32** 1541
Hioe F T and Carroll C E 1985 *J. Opt. Soc. Am.* B **2** 497
[21] Radmore P M 1982 *Phys. Lett.* A **87** 285
[22] Lee H-W and George T F 1984 *Phys. Rev.* A **29** 2509
[23] Zakrzewski J 1985 *Phys. Rev.* A **32** 3748
[24] Hioe F T and Carroll C E 1986 *J. Phys. A: Math. Gen.* **19** 3579
[25] Ishkhanyan A M 1988 *Izv. Akad. Nauk Armenii Fiz.* **23** 190
[26] Prants S V 1990 *Phys. Lett.* A **144** 225
[27] Vitanov N V and Garraway B M 1996 *Phys. Rev.* A **53** 4288
[28] Ishkhanyan A M 1996 *J. Contemp. Phys. (Armenian Acad. Sci.)* **31** 10
[29] Ishkhanyan A M 1997 *J. Phys. A: Math. Gen.* **30** 1203
[30] Ishkhanyan A M 2000 *Opt. Commun.* **176** 155
[31] Carroll C E and Hioe F T 1986 *J. Phys. A: Math. Gen.* **19** 1151
[32] Carroll C E and Hioe F T 1987 *Phys. Rev.* A **36** 724
[33] Ishkhanyan A M and Melikdzanov D Yu 1988 *Dokl. Akad. Nauk Armenii* **86** 71
[34] Carroll C E and Hioe F T 1990 *Phys. Rev.* A **42** 1522
[35] Carroll C E and Hioe F T 1992 *Phys. Rev. Lett.* **68** 3523
[36] Ishkhanyan A M 1994 *Izv. Akad. Nauk Armenii Fiz.* **29** 9
[37] Laine T A and Stenholm S 1996 *Phys. Rev.* A **53** 2501
[38] Vitanov N V and Stenholm S 1997 *Phys. Rev.* A **55** 648
[39] Kamke E 1976 *Handbook of Ordinary Differential Equations* (Moscow: Nauka)
[40] Erdelyi A, Magnus W, Oberhettinger F and Tricomi F G 1953 *Higher Transcendental Functions* (New York: McGraw-Hill)
Slater L J 1966 *Generalized Hypergeometric Functions* (Cambridge: Cambridge University Press)
Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
[41] Ishkhanyan A M *Phys. Rev.* A, to be published